Mean-field theories of random advection

Alan R. Kerstein
Combustion Research Facility, Sandia National Laboratories, Livermore, California 94551-0969

Patrick A. McMurtry
Department of Mechanical Engineering, University of Utah, Salt Lake City, Utah 84112
(Received 15 September 1993)

Two mean-field theories of random advection are formulated for the purpose of predicting the probability density function (PDF) of a randomly advected passive scalar, subject to an imposed mean scalar gradient. One theory is a generalization of the mean-field analysis used by Holzer and Pumir [Phys. Rev. E **47**, 202 (1993)] to derive the phenomenological model of Pumir, Shraiman, and Siggia [Phys. Rev. Lett. **66**, 2984 (1991)] governing PDF shape in the imposed-gradient configuration. The other theory involves a Langevin equation representing concentration time history within a fluid element. Predicted PDF shapes are compared to results of advection simulations by Holzer and Pumir. Both theories reproduce gross trends, but the Langevin theory provides the better representation of detailed features of the PDF's. An analogy is noted between the two theories and two widely used engineering models of turbulent mixing.

PACS number(s): 47.27.Gs, 05.40.+j, 02.50.−r

I. INTRODUCTION

Motivated by experimental observations that the probability density function (PDF) of a passive scalar advected by turbulence has long tails in some instances, Pumir, Shraiman, and Siggia (PSS) proposed that non-Gaussian behavior is an inherent property of random advection [1]. They formulated a phenomenological model of the steady-state statistics of a passive scalar subject to a mean gradient, undergoing random advection and molecular diffusion. The model predicts that the PDF of the scalar deviation from its mean value has exponential tails.

For this configuration, the occurrence of tails that are exponential, or nearly so, has been confirmed experimentally [2,3] and numerically [4–6]. Holzer and Pumir (HP) provided further mathematical interpretation based on exact and mean-field analyses of stochastic processes that emulate the advection of a passive diffusive scalar [5]. Their mean-field theory recovers the phenomenological PDF evolution equation postulated by PSS and identifies the approximations subsumed in the equation. They note that the Gaussian core and exponential tails predicted by the phenomenological equation agree with PDF properties determined by numerical simulation of advection diffusion. The agreement supports but does not unequivocally confirm the validity of the mean-field theory.

Here, it is shown that a key assumption within the HP mean-field theory can be avoided. To obtain the evolution equation of PSS from their mean-field formulation, HP assume that the effects of advection and diffusion are additive. This assumption is avoidable because their formulation at the point of introduction of this assumption is exactly solvable. The solution is obtained and is compared to the PSS result. The present solution exhibits a dependence of PDF shape on the statistics of the advection process that is absent from the PSS result. The PSS result is recovered as a special case. In effect, a generalized HP theory is obtained.

The validity of the generalized theory is not assured because the HP formulation involves other assumptions of undetermined accuracy. Alternative assumptions are introduced that yield a linear Langevin equation governing the time history of scalar evolution within a fluid element. This formulation is mathematically simpler and has a clearer physical interpretation. Numerical results reproduce the salient features of PDF's obtained by numerical simulation of advection diffusion [5], and yield good quantitative agreement in some instances. In contrast, generalized HP theory captures only gross trends.

Present application of the two mean-field theories is limited to advection consisting of a time sequence of statistically independent jump events, formulated within the framework of the linear-eddy model of advection diffusion [4,5]. For advection processes of this type, the theories predict Poisson-like PDF tails, consistent with previous theoretical and numerical results [5]. It has been noted that the mechanism causing the Poisson-like tails is specific to jump processes, and that continuum flow generates long-tailed PDF's by a different mechanism [7]. The Langevin formalism can accommodate the additional mechanism, but this extension is not implemented here.

It is shown that both theories can be represented as weighted sums of random increments, a result that is useful both for numerical work and for interpretation of qualitative trends. This representation can be interpreted physically in the Langevin framework, but is purely formal for generalized HP theory. Moreover, it is noted that a recently derived expression for the pressure PDF in a Gaussian flow [8] is formally equivalent to a sum of independent increments, though as in the generalized HP theory, this equivalence has no obvious physical interpretation. The formal result is nevertheless intriguing because the representation is exact in this case, so its mathematical structure cannot be attributed to modeling assumptions. It is therefore interesting to speculate about the possible generality of this representation.

Another open question is whether the mean-field formulations presented here constitute the totality of possible mean-field theories of random advection. In this regard, it should be noted that a stochastic model with some attributes of random advection has been studied using exact, mean-field, and numerical methods [9], yielding results qualitatively consistent with the results presented here. However, that model does not conform to a strict mechanistic definition of random advection diffusion as stochastic evolution governed by two distinct processes: advection, which moves fluid elements without changing their internal state (i.e., scalar value), and diffusion, which causes internal state changes through local interactions between fluid elements. Models not conforming to this definition may have properties that do not correspond to advection diffusion. Nevertheless, they may be useful for purposes of refining physical intuition.

The possibility of formulating new or improved mean-field theories is of practical as well as conceptual interest because the two theories developed here are formally analogous to two of the most widely used engineering models of turbulent mixing, the coalescence-dispersion (CD) and interaction by exchange-with-the-mean (IEM) models [10]. Mean-field analysis may provide insight into the performance of these models and may lead to improved formulations.

This paper is organized as follows. In Sec. II, the linear-eddy model formulation analyzed and numerically implemented by HP is outlined. In Secs. III and IV, the generalized HP and Langevin mean-field theories, respectively, are constructed, analyzed, and compared to the numerical results of HP. Implications of the analysis and numerical comparisons are summarized in Sec. V.

II. LINEAR-EDDY MODEL

In the context of fluid flow in two or more dimensions, random advection diffusion as defined in Sec. I corresponds to evolution of a scalar field $\Theta(\mathbf{x},t)$ governed by the equation

$$\frac{\partial\Theta}{\partial t} + \mathbf{v}\cdot\nabla\Theta = \kappa\nabla^2\Theta, \qquad (1)$$

where κ is the molecular diffusivity. The velocity field $\mathbf{v}(\mathbf{x},t)$ is taken to be a realization of a given random process, unaffected by $\Theta(\mathbf{x},t)$ (i.e., Θ is a passive scalar).

Advection diffusion encompasses a broader class of processes than those governed by Eq. (1). The advective term in Eq. (1) represents motion along continuous trajectories. However, advection processes involving discontinuous jumps are also of interest, both in their own right and as idealized representations of continuum flow. For example, the Lévy walk is a jump process whose scaling properties have proven useful in interpreting turbulent transport phenomena [11].

The linear-eddy model of random advection diffusion, involving discontinuous jumps, has been introduced for the purpose of emulating the physics of Eq. (1) on a one-dimensional domain [4]. In this model, advection is implemented as instantaneous events whose time sequence is governed by Poisson statistics (i.e., the epochs of the events are statistically independent). Each event is a spatial rearrangement of the scalar field within a randomly selected interval of the domain. Each point within the interval, except the midpoint, undergoes a finite displacement, i.e., a jump. An important feature of the linear-eddy model is that each event involves displacement of a set of points, rather than a single point. Unlike single-point jump processes, the linear-eddy model thus incorporates multipoint correlations reflecting the spatially correlated motion that occurs in continuum flow. A rearrangement rule reflecting features of vortical motion incorporates additional continuum-flow phenomenology [4].

For present purposes, a variant of the linear-eddy model formulated by HP is adopted. HP interpret their formulation in terms of turbulent mixing processes. Here, it is treated as an advection process in its own right. The present objectives are to formulate mean-field theories of the linear-eddy advection diffusion process, and to evaluate these theories by comparing predictions to numerical results of HP. The mean-field theories are formally applicable to continuum flow, indicating the possible generality of the present approach.

HP consider advection diffusion in one and two dimensions (1D and 2D). Space is discretized into equally spaced points in 1D and sites of a square lattice in 2D. Various rearrangement rules are employed, several of which are specified in Secs. III and IV. For illustrative purposes, the simplest 1D formulation is considered. The simplest rearrangement rule consists of random selection of a point n, followed by an exchange ("flip") of the scalar values at points n and $n+1$, i.e., the transformation

$$(\Theta_n,\Theta_{n+1}) \to (\Theta_{n+1},\Theta_n). \qquad (2)$$

The advection process, consisting of a random time sequence of flips, is the linear-eddy analog of the advection term in Eq. (1). The diffusion process is implemented by numerical solution of Eq. (1), without the advection term, on the discretized domain. The numerical solution involves time integration during the time interval between successive flips, followed by implementation of the next flip, followed by further time integration, and so forth.

This formulation is used by HP to compute the steady-state statistics of the imposed-scalar-gradient configuration. The scalar field is assigned an initial value $\Theta_n(0) = n$. By symmetry, the mean scalar field at any time $t>0$ is $\langle\Theta_n(t)\rangle = \Theta_n(0)$. After an initial transient, the scalar field relaxes to a statistically steady state. For large t, the PDF of the scalar deviation $c_n(t) = \Theta_n(t) - \Theta_n(0)$ does not depend on n or t, and therefore is denoted $f(c)$. The PDF depends on a single parameter, the ratio $K = \tau_f/\tau_e$

of time scales τ_f and τ_e governing advection and diffusion, respectively. For the linear-eddy formulation based on Eq. (2), HP take τ_f to be the mean time between flips involving the same pair of points, and they take τ_e to be a^2/κ, where a is the lattice spacing, normalized to unity. More general definitions of the time scales are given in Sec. IV.

Mean-field theories of advection diffusion are developed with reference to this illustrative case. Generalization to other cases is considered subsequently.

III. GENERALIZATION OF HP MEAN-FIELD THEORY

HP develop a mean-field theory of the advection diffusion process of Sec. II in two parts. Molecular diffusion is analyzed starting from the evolution equation for the scalar value Θ_i in cell i of the discretized domain. In contrast, advection is analyzed starting from an equation for the evolution of the PDF of this quantity, $f(\Theta_i)$, and coarse graining to obtain a Fokker-Planck equation. These separate analyses are subsequently combined by summing the two effects to obtain an overall evolution equation for the Fourier transform of $f(\Theta_i)$. There is no obvious justification for the assumed additivity of effects.

Here, the summation of effects at the coarse-grained level is avoided by incorporating the effect of advection into the fine-grained description of the diffusion process. Namely, the starting point is taken to be

$$c_i(t) = (1-2\epsilon)c_i(0) + \epsilon[c_{i-1}(0) + c_{i+1}(0)] + \eta. \quad (3)$$

Here, η is a random forcing term representing advective effects; its properties are specified shortly. Apart from this term, Eq. (3) is the finite difference representation of the diffusion equation, Eq. (1), with the advective term omitted, where $\epsilon = \kappa t/a^2$. This representation is the starting point of the HP mean-field analysis of diffusion effects, except that it is formulated here in terms of the scalar deviation $c_i = \Theta_i - i$ rather than Θ_i. The time argument zero on the right-hand side of Eq. (3) refers not to the initial condition, but to an arbitrary reference time in the statistically steady state.

For t small enough so that $\epsilon \ll 1$, Eq. (3) is a quantitatively accurate approximation of the advection diffusion process of Sec. II. Here, Eq. (3) is analyzed for arbitrary ϵ. Though the small-ϵ limit is encompassed by the analysis, the results are not necessarily more valid in this limit owing to the following modeling assumption that is adopted.

As in the HP analysis, it is assumed that the random variables $c_j(0)$ ($j = i-1,\ i,\ i+1$) are statistically independent. This assumption is ostensibly most accurate in the limit $\tau_e \gg \tau_f$, i.e., the regime in which the decorrelating effect of flips dominates the correlating effect of diffusion. However, relaxation to a statistically steady state implies that, for any τ_f/τ_e, Θ differences between neighboring points grow until diffusive effects balance advective effects. For present purposes, the statistical independence assumption is regarded as an approximation of undetermined accuracy.

In the same vein, it is assumed here that the quantities $c_j(0)$ are independent of the random forcing term η. The interpretation of η as an advection term follows from recasting Eq. (3) as an equation for the time history of c in a comoving reference frame, or in fluid mechanical terminology, the reference frame attached to a fluid element. In that frame, a jump one step to the left or right leaves the scalar value $\Theta(t)$ within the fluid element unchanged, but changes the scalar deviation $c(t) = \Theta(t) - i(t)$ by $+1$ or -1, respectively. Here, location i is parametrized by t to represent the time history of fluid element location subject to the jump process.

Consistent with this viewpoint, Eq. (3) is rewritten as

$$c_0(t) = (1-2\epsilon)c_0(0) + \epsilon[c_-(0) + c_+(0)] + \eta(t), \quad (4)$$

where the subscripts 0, $-$, and $+$ respectively denote the current location of the fluid element and its left and right neighbors. Here, $\eta(t)$ is a random variable representing the jump-induced displacement of the fluid element during the time interval $[0,t]$. [More precisely, $\eta(t)$ is the negative of this displacement, but for advection processes considered here, symmetry allows the sign change to be omitted.] For the advection process of Sec. II, the PDF $g[\eta(t)]$ of the random variable $\eta(t)$ can be derived exactly. Before considering this case, general features of the relationship between g and the statistics of c_0 are examined.

As noted in Sec. II, the scalar deviation PDF is independent of spatial location and time under steady-state conditions in the imposed-uniform-gradient configuration. Therefore, $c_0(t)$, $c_+(0)$, and $c_-(0)$ are governed by the PDF $f(c)$. Under the statistical independence hypotheses that have been adopted, Eq. (4) determines $f(c)$ for given ϵ and $g(\eta)$ without further approximation.

The solution is most readily obtained by invoking properties of the characteristic function of a random variable, defined as the Fourier transform of the PDF,

$$\phi_z(k) = \int_{-\infty}^{\infty} e^{ikZ} h(Z)\, dZ, \quad (5)$$

where h is the PDF of a random variable z and ϕ_z is the characteristic function of z. The key property of interest is the following relationship [12] between the characteristic function ϕ_λ of a weighted sum $\lambda = \sum_{i=1}^{n} w_i z_i$ of independent random variables z_i and the characteristic functions ϕ_i of z_i, $i = 1, \ldots, n$:

$$\phi_\lambda(k) = \prod_{i=1}^{n} \phi_i(w_i k). \quad (6)$$

For the special case $n = 2$, $w_1 = w_2 = 1$, this relation is obtained by Fourier transforming the relation $h_s(S) = \int_{-\infty}^{\infty} h_1(u) h_2(S-u)\, du$, where h_s is the PDF of the random variable $s = z_1 + z_2$, h_1 is the PDF of z_1, and h_2 is the PDF of z_2. Generalization to an arbitrary set of weights w_i is straightforward. The PDF of λ is obtained by taking the inverse Fourier transform of Eq. (6).

HP applied Eq. (6) to Eq. (3) without the η term. Applying it here to Eq. (4), the relation

$$\phi_c(k) = \phi_c[k(1-2\epsilon)]\phi_c^2(k\epsilon)\phi_\eta(k) \tag{7}$$

is obtained. It is convenient to analyze the logarithm of this equation, involving sums instead of products. The solution is

$$\ln[\phi_c(k)] = \sum_{n=0}^{\infty}\sum_{p=0}^{n} 2^p \binom{n}{p} \ln\{\phi_\eta[k\epsilon^p(1-2\epsilon)^{n-p}]\}, \tag{8}$$

which is readily verified by substitution into Eq. (7).

In the limit $\epsilon \to 0$, the only nonvanishing term of Eq. (8) is $n = p = 0$, giving $\phi_c(k) = \phi_\eta(k)$. This is a trivial result because $\eta(t)$ is identically zero for $t = 0$. As noted earlier, the validity of Eq. (3) in this limit does not imply the validity of the small-ϵ limit of Eq. (8), owing to the statistical assumptions that have been adopted.

For finite ϵ, the dependence of $f(c)$ on the functional form of $g(\eta)$ is examined. (Here, the argument t is suppressed.) For the jump process under consideration, g is a discrete distribution. It can be approximated by a continuous distribution if its variance is much greater than unity, as occurs in some cases of interest. For the moment, distributions g are considered without reference to the underlying advection process.

The choice $g(\eta) = \frac{1}{2\sqrt{D}}\exp(-|\eta|/\sqrt{D})$, the two-sided exponential distribution, is of particular interest. Substitution of its characteristic function, $\phi_\eta(k) = \frac{1}{1+Dk^2}$, into Eq. (8) recovers the solution obtained by HP. [The simpler solution obtained originally by PSS for $\epsilon = \frac{1}{2}$ is recovered by substituting this $\phi_\eta(k)$ into Eq. (9), below.] As noted in Sec. I, the result of HP was obtained by assuming the additivity of advection and diffusion effects in the framework of mean-field theory. Thus, the additivity assumption is equivalent to selection of a particular functional form of $g(\eta)$.

HP adopted the additivity assumption for the specific purpose of reproducing the PSS result, and noted that it has no obvious physical basis. Equation (8) indicates that mean-field theory admits other possibilities. For example, if $g(\eta)$ is Gaussian with variance σ^2, then $\phi_\eta(k) = \exp(-\sigma^2k^2/2)$, giving Gaussian $f(c)$. Thus, mean-field theory *per se* does not require $f(c)$ to be long tailed. The exponential tails deduced from mean-field theory by PSS and HP are due to the functional form of $g(\eta)$ imposed by the additivity assumption.

For advection processes involving independent jump events, long tails nevertheless occur owing to the Poisson statistics governing fluctuations of the time interval between successive events. This has been demonstrated by exact analysis of a simplified advection model [5], and is recognized as a general property of jump-type advection processes [7]. It also follows from mean-field theory based on an exact representation of jump statistics, as demonstrated shortly.

Further analysis of Eq. (8) is restricted to the jump process of Sec. II. For this purpose, ϵ is set equal to $\frac{1}{2}$. As noted by HP, there is no uniquely preferred choice of ϵ, and there should be no significant qualitative differences between results for different values. The value $\epsilon = \frac{1}{2}$ is chosen because it allows considerable simplification of Eq. (8), namely

$$\phi_c(k) = \prod_{n=0}^{\infty} [\phi_\eta(2^{-n}k)]^{2^n}. \tag{9}$$

A useful interpretation of Eq. (9) is obtained by comparing it to Eq. (6), which indicates that $f(c)$ is the PDF of the random variable c obtained by forming the following weighted sum of random samples $\eta_{i,n}$ from the PDF $g(\eta)$:

$$c = \sum_{n=0}^{\infty} 2^{-n} \sum_{i=1}^{2^n} \eta_{i,n}. \tag{10}$$

The random samples $\eta_{i,n}$ are statistically independent; double indexing is adopted for convenience.

This representation of c is useful for analysis and computations, but it has no obvious physical interpretation. The mean-field theory developed in Sec. IV likewise yields a weighted sum of random samples from the advection PDF, with a straightforward physical interpretation in that instance. As noted in Sec. I, a recently derived expression for the pressure PDF in a Gaussian flow [8] involves an infinite product representation of the characteristic function, indicating that the statistics are based on a weighted sum of independent random variables. This representation may thus be more broadly applicable than is apparent from the derivation of Eq. (10).

Equation (10) is now applied to the advection process of Sec. II. Recall that $\eta(t)$ in Eq. (4) is a random variable representing the jump-induced displacement of the fluid element during the time interval $[0, t]$. For the choice $\epsilon = \frac{1}{2}$, $t = a^2/(2\kappa) = \tau_e/2$. In Sec. II, the mean time between flips involving the same pair of points is denoted τ_f. Since a given point can flip with either neighbor, the mean time between displacements of a point is $\tau_f/2$. Therefore the mean number of displacements during $[0, t]$, here denoted by μ, is $\tau_e/\tau_f = K^{-1}$. The dependence of $f(c)$ on the parameter μ is considered.

Because displacements are statistically independent events, the number of displacements during $[0, t]$ is a random sample from the Poisson distribution with mean μ. Moreover, positive and negative displacements are independent, so the net displacement can be represented as $\eta(t) = \eta^+(t) - \eta^-(t)$, where the number of positive and negative displacements, $\eta^+(t)$ and $\eta^-(t)$, respectively, are each randomly sampled from the Poisson distribution with mean $\mu/2$.

It is convenient to adopt this representation in Eq. (10). Each $\eta_{i,n}$ is expressed as $\eta_{i,n} = \eta_{i,n}^+ - \eta_{i,n}^-$, where each $\eta_{i,n}^\pm$ is Poisson distributed with mean $\mu/2$. To simplify the notation, Eq. (10) is expressed in the equivalent form

$$c = s^+ - s^-, \tag{11}$$

where each of the random variables $s^\pm$ is a weighted sum of random variables as given by Eq. (10) with corresponding superscripts on $\eta_{i,n}$. In this formulation, the individual terms in each sum are governed by the Poisson distribution [12]

$$g_{\mu/2}(\eta) = e^{-\mu/2} \sum_{j=0}^{\infty} \frac{(\mu/2)^j}{j!} \delta(\eta - j), \tag{12}$$

where η denotes any of the random variables $\eta^{\pm}_{i,n}$. The random variables s^+ and s^- are identically distributed, with PDF denoted $h(s)$.

This formulation is adopted because it is convenient in numerical work to determine $h(s)$ based on Eq. (12), and then to determine $f(c)$ by convolving h with itself, as prescribed by Eq. (11). In contrast, the distribution of net displacements involves a double sum which is far more costly to evaluate.

Based on the characteristic function of Eq. (12), Eq. (11) can be used to solve Eq. (9), yielding

$$\phi_c(k) = \exp\left[\mu \sum_{j=1}^{\infty} \frac{(ik)^{2j}}{(1-2^{1-2j})(2j)!}\right]. \qquad (13)$$

Though this expression is convenient for certain analytical purposes, it is not used here.

The formulation based on Eq. (11) provides a transparent demonstration of the origin of long-tailed PDF's in the case of jump-type advection diffusion processes. The random variable c is represented as a weighted sum of Poisson-distributed random variables. The tails of $f(c)$ are therefore of the same form as the tails of the PDF $g_{\mu/2}(\eta)$ governing the individual terms. In Eq. (12), adoption of Stirling's approximation for $j!$ gives, for $\eta \gg 1$, $\ln g_{\mu/2}(\eta) \sim -\eta \ln \eta$.

Other aspects of PDF shape are sensitive to the parameter μ. The mean and standard deviation of $g_{\mu/2}(\eta)$ are $\mu/2$ and $(\mu/2)^{1/2}$, respectively. For $\mu \gg 1$, the discreteness of the distribution is effectively a fine-grain property, so $f(c)$ is relatively insensitive to details of the advection process.

Formal derivations of these properties are omitted because the underlying physics is understood [5,7], and because numerical demonstrations are provided. Numerical work is facilitated by the following reformulation of Eq. (10). In applying this relation to $s^{\pm}$, the summation over i involves, for given n, 2^n random variables governed by the PDF $g_{\mu/2}$. Physically, this sum represents the number of displacements of a given sign during a time interval $[0, 2^n t]$, so the sum is governed by the PDF $g_{2^{n-1}\mu}$, i.e., the Poisson distribution with mean $2^{n-1}\mu$. (This is the "reproductive property" of the Poisson distribution [12].) Therefore the sum over i is replaced by a random variable η_n governed by $g_{2^{n-1}\mu}$. To account for the weighting 2^{-n} in the sum over n, a transformation to the random variable $\zeta_n = 2^{-n}\eta_n$ is performed in Eq. (12), giving

$$g_{2^{n-1}\mu}(\zeta_n) = e^{-2^{n-1}\mu} \sum_{j=0}^{\infty} \frac{(2^{n-1}\mu)^j}{j!} \delta(\zeta_n - 2^{-n}j). \quad (14)$$

The PDF $h(s)$ is determined for given μ by convolving $g_{\mu/2}(\zeta_0)$ with $g_{\mu}(\zeta_1)$, convolving the result with $g_{2\mu}(\zeta_2)$, and so forth, until convergence. For μ values considered here, the number of iterations required is of order 10. Finally, $h(s)$ is convolved with itself to obtain $f(c)$.

Numerical results are compared in Fig. 1 to the simulation results of HP for the advection diffusion process of Sec. II. The mean-field PDF's reflect the qualitative trends deduced from the properties of $g_{\mu/2}$. For small μ (large K), the simulation results exhibit features sensitive to details of the advection process. The origin of these features is discussed in Sec. IV. It is evident that the mean-field theory has little predictive value with regard to features of PDF shape other than overall trends.

Also shown in the figure are results based on an alter-

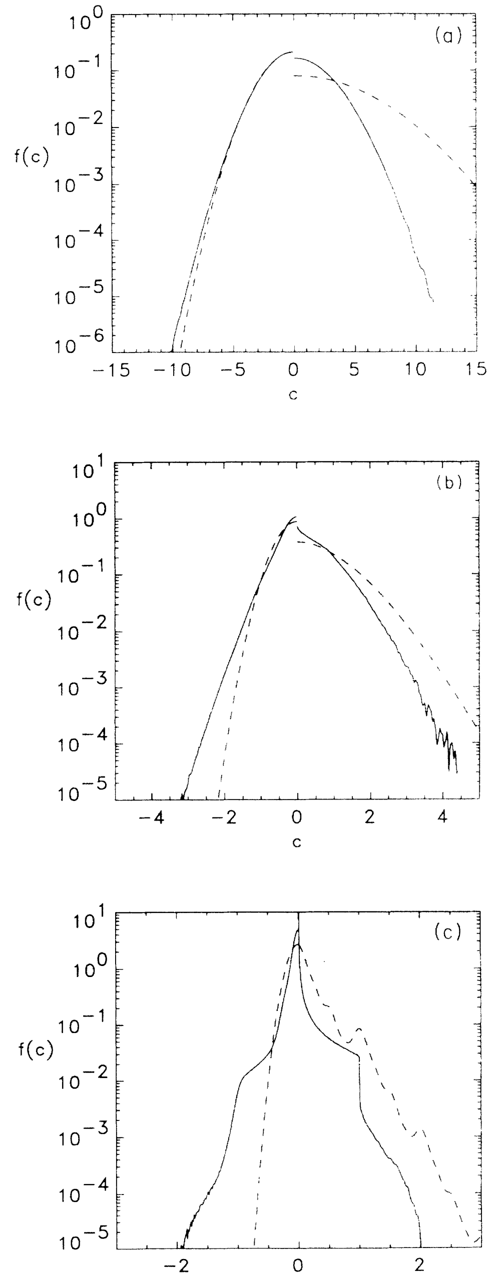

FIG. 1. Semilogarithmic plot of the PDF of scalar deviation c for advection diffusion in 1D based on the two-point flip. Left of center line: simulation results of HP [5] (solid); Gaussian of the same variance (dashed). Right of center line: predictions of generalized HP theory (dashed) and Langevin theory (solid). Results are parametrized by the ratio K of advection to diffusion time scales. (a) $K = 0.083$; (b) $K = 1.66$; (c) $K = 16.6$.

nate mean-field theory, which appears to be more promising with regard to quantitative prediction. The formulation and application of this theory are considered next.

IV. LANGEVIN MEAN-FIELD THEORY

Equation (8) determining the statistics of c in terms of ϵ and the statistics of η was obtained from Eq. (4) by assuming that $c_0(0)$, $c_+(0)$, and $c_-(0)$ are independent random variables. A different mean-field theory is obtained by setting $c_+(0)$, and $c_-(0)$ equal to zero. In fact, this is more in keeping with the mean-field concept because it is a replacement of these quantities by their mean values.

The key practical distinction between this formulation and generalized HP theory is that Eq. (4) now yields a meaningful result in the limit $\epsilon \to 0$. Recalling that $\epsilon = \kappa t/a^2$, Eq. (4) reduces in this limit to the linear Langevin equation

$$\frac{dc}{dt} = -\beta c + \eta'(t), \tag{15}$$

where the subscript and argument of c are suppressed, $\beta = 2\kappa/a^2$, and $\eta'(t) = \frac{d\eta(t)}{dt}$. The solution of this equation for any realization of the random process $\eta'(t)$ is

$$c(t) = \int_{-\infty}^{t} \eta'(t')e^{-\int_{t'}^{t}\beta(t'')\,dt''}\,dt'$$
$$= \int_{-\infty}^{t} \eta'(t')e^{-(t-t')\beta}\,dt'. \tag{16}$$

The solution is shown for the more general case of time-varying β as well as for constant β.

The purpose of the generalization is to indicate features of advection diffusion that are not represented in the mean-field analysis but might be captured by a more general formulation. The most general formulation in the framework of Eq. (15) is obtained by taking $\beta(t)$ to be a random process that may depend on $c(t')$ and $\eta'(t')$ for all $t' \le t$. If the statistics of $\beta(t)$, including these dependences, were known exactly, then Eq. (15) would be an exact representation of the concentration time history within a fluid element. If the exact representation involved dependence of β on c, then Eq. (15) would be nonlinear in c and Eq. (16) would no longer apply.

A recent analysis of continuum advection implies dependence of β on c or η'. For the case of Gaussian random flow, the functional form $f(c) \sim |c|^{-1/2}\exp(-\gamma|c|)$ was obtained for $|c| \gg 1$ [7]. This result was attributed to the fluctuation statistics of a decay parameter analogous to β. It was noted that non-Gaussian statistics of the displacement process [here, $\eta'(t)$] are not required to obtain this result.

Applying Eq. (15) to continuum flow, it is evident from Eq. (16) that allowing β to be time dependent, but independent of c and η', does not give non-Gaussian $f(c)$ for Gaussian $\eta'(t)$. Dependence of β on c or η' is also required.

This generalization is not implemented here. Rather, Eq. (15) with constant β is applied to jump-type advection processes. It was noted earlier that jump-type advection, reflected in the statistics of $\eta'(t)$, causes the tails of $f(c)$ to be Poisson-like. [This behavior is captured by Eq. (15) with constant β. To see this, integrate over a time interval much shorter than $1/\beta$ and note that the tails of c are determined by the Poisson statistics of jump events, as in generalized HP theory.] This observation does not rule out the possibility that mechanisms omitted from the model, such as time variation of β, may induce comparable, or slower, large-c falloff. To assess the validity of the present formulation, and to motivate future theoretical development (with possible relevance to continuum advection), the Langevin theory is compared to numerical simulation results of HP.

For the advection diffusion process of Sec. II, $\eta'(t)$ can be expressed in the form

$$\eta'(t) = \sum_{j=1}^{\infty} \eta_j \delta(t - t_j), \tag{17}$$

where $t_j < t$ is the epoch of the jth displacement event, ordered backward in time from time t, and the displacement η_j is a Bernoulli trial (i.e., ± 1 with equal probability). In accordance with Poisson statistics, the separations $s_j = t_{j-1} - t_j$ are independent random samples from the PDF $h(s) = (1/T)\exp(-s/T)$, where the mean time T between events is $\tau_f/2$, as in Sec. III. (In this notation, t_0 corresponds to t.) In terms of these quantities, Eq. (16) gives

$$c = \eta_1 e^{-s_1\beta} + \eta_2 e^{-(s_1+s_2)\beta} + \eta_3 e^{-(s_1+s_2+s_3)\beta} + \cdots . \tag{18}$$

As in generalized HP theory, the treatment of the advection process is exact. Rescaling time in units of $1/\beta$ and noting that $\beta = 2/\tau_e$, the Langevin theory is parametrized by $\beta T = K$.

Here, c is expressed as a sum of random variables, but the terms in the sum are not statistically independent because the jth term involves quantities $s_1, \ldots, s_{j-1}$ that also appear in the preceding terms. Therefore the analytical techniques of Sec. III are not applicable. To obtain a sample value of c from Eq. (18), a Monte Carlo technique is used, involving random sampling of η_j and s_j from their respective PDF's and summing a sufficient number of terms (generally within the range 10^2–10^3) to obtain convergence of the sample value.

Results are compared to the numerical simulations of HP and to generalized HP mean-field theory in Fig. 1. The statistical variability of the Langevin-theory results reflects the finite number of c values (generally within the range 10^6–10^7) used to form the histogram of $f(c)$. It is evident that the Langevin theory is in better overall agreement with simulated PDF's than is generalized HP theory, especially at large K. Generalized HP theory provides a better qualitative representation only at small c. The central peaks predicted by the Langevin theory are too sharp because the exponential decay of c within the theory is too rapid at small c. This reflects the assumed decay toward $c = 0$ rather than toward the c

value corresponding to the local environment of the fluid element.

For large K, PDF shape is sensitive to details of the advection process, as noted by HP. In this regime, the epoch of the most recent displacement is the dominant influence, and its manifestation is the shoulder on each side of the central peak. Additional far-tail structure reflects rare occurrences of multiple displacements within a time interval much shorter than T.

As noted earlier, Poisson-like tails of $f(c)$ can be inferred from Eq. (16). The dependence of the onset of this behavior on K is indicated by the structure of Eq. (18). For small K, the exponentials multiplying the terms η_j fall off gradually, so c is effectively the sum of a large number of nearly identical random variables. This explains the nearly Gaussian core of $f(c)$ for small K. For larger K, this approximate conformance to the central limit theorem no longer holds, resulting in strongly non-Gaussian PDF's.

For further evaluation of the Langevin theory, it is applied to two other advection diffusion processes simulated by HP in 1D and 2D, respectively. The 1D process is motivated by the spatial discretization effects apparent in the shape of the $K = 16.6$ PDF of Fig. 1. To suppress discretization effects, HP simulated advection diffusion based on the multipoint flip

$$(\Theta_{n-l}, \Theta_{n-l+1}, \ldots, \Theta_{n+l-1}, \Theta_{n+l}) \rightarrow (\Theta_{n+l}, \Theta_{n+l-1}, \ldots, \Theta_{n-l+1}, \Theta_{n-l}), \qquad (19)$$

where flip "radius" l is a random variable sampled from the half-Gaussian PDF

$$p(l) = \left(\frac{2}{\pi l_0^2}\right)^{1/2} \exp[-\tfrac{1}{2}(l/l_0)^2], \qquad l > 0. \qquad (20)$$

The parameter l_0 was assigned the value $16a$, large enough so that lattice discretization would not significantly affect the shape of $f(c)$. Therefore Eq. (20) and the ensuing mean-field analysis of this case are formulated for a spatial continuum $-\infty \leq x \leq +\infty$, with distance expressed in units of a.

To represent this process within the Langevin theory, the quantity η_j in Eq. (18), representing the displacement at time t_j, is taken to be a random sample from a cumulative distribution function (CDF) $G(\eta)$ determined from $p(l)$. The displacement CDF $G(\eta)$ is the ratio of the frequency $\Phi(\eta)$ of displacements in the range $[-\infty, \eta]$ to the total displacement frequency $\Phi(\infty)$. These quantities are derived in terms of the flip frequency per unit distance, denoted ρ. (In lattice terminology, ρa is the flip frequency for given n. The continuum analog of the flip center n is denoted x_0.)

Denote the current location of the fluid element as $x = 0$. Then $\Phi(\eta)$ is the frequency of all flip events whose centers x_0 are to the left of $x = \eta/2$ (otherwise the displacement would exceed η) and whose radius l exceeds $|x_0|$ (otherwise the flip would not include the point $x = 0$). Thus,

$$\Phi(\eta) = \rho \int_{-\infty}^{\eta/2} dx_0 \int_{|x_0|}^{\infty} p(l)\, dl. \qquad (21)$$

Substitution of Eq. (20) for $p(l)$ gives

$$\Phi(\infty) = \left(\frac{8}{\pi}\right)^{1/2} \rho l_0 \qquad (22)$$

and

$$G(\eta) = 1 + \left(\frac{\pi}{8}\right)^{1/2} \frac{\eta}{2l_0} \left[1 - \operatorname{erf}\left(\frac{\eta}{\sqrt{8} l_0}\right)\right] - \tfrac{1}{2} \exp[-\tfrac{1}{8}(\eta/l_0)^2]\,, \quad \eta \geq 0. \qquad (23)$$

Because the PDF $g(\eta)$ is an even function of η, $G(\eta)$ for $\eta < 0$ is determined from $G(-\eta) = 1 - G(\eta)$.

Equations (22) and (23), in conjunction with the statistical properties of the event separations s_j, constitute an exact representation of the advection process on the spatial continuum. (An exact lattice representation can also be obtained, but as noted, the numerical results would not be significantly different.) The final step in the formulation of the Langevin theory for this process is specification of the time-scale ratio K in terms of parameters of the theory.

HP adopt the general definitions $\tau_f = (\rho \xi^d)^{-1}$ and $\tau_e = \xi^2/\kappa$ for the d-dimensional advection and diffusion time scales, respectively, where ξ is a characteristic advection length scale. For the advection process governed by Eq. (20), they take $\xi = 2\langle l \rangle = (8/\pi)^{1/2} l_0$, giving $\tau_f = \Phi^{-1}(\infty)$.

The evaluation of β in Eq. (18) is based on the relation $\beta = 2\kappa/L^2$, where L is a length scale over which diffusion acts. For the advection process consisting of two-point flips, $L = a$ because the flips generate fluctuations on the scale of the lattice spacing. For advection governed by Eq. (20), the lattice spacing is no longer the relevant length scale. In accordance with HP, L as well as ξ is set equal to $2\langle l \rangle$. Though there is some arbitrariness in this assignment, computations involving variation of L over a factor of two indicate that qualitative comparisons are not affected by the length scale assignment. The choice $L = \xi$ gives $\beta = 2/\tau_e$ and thus $K = \beta/[2\Phi(\infty)]$.

Langevin theory results are compared to simulation results of HP for this process in Fig. 2. It is evident that Langevin theory captures the dependence of tail shape on K. As in Fig. 1, the central peaks are too sharp.

A 2D advection diffusion process simulated by HP on a square lattice involves a 12-site flip, illustrated in Fig. 3. The 12 sites closest to the center of a selected lattice cell are permuted by a 90° clockwise or counterclockwise rotation. Denoting the lattice directions as i and j, the initial scalar field is $\Theta_{i,j}(0) = i$. The 2D analog of Eq. (3) is

$$c_{i,j}(t) = (1 - 4\epsilon)c_{i,j}(0) + \epsilon[c_{i-1,j}(0) + c_{i+1,j}(0) + c_{i,j-1}(0) + c_{i,j+1}(0)] + \eta, \qquad (24)$$

where again, $\epsilon = \kappa t/a^2$. Invoking the mean-field approximation, the quantity in square brackets is set equal to zero. Because the mean scalar value changes only in the direction i, η is now the i component of the flip-induced displacement. Equations (15)–(18) of Langevin theory apply, with parameter assignments and statistical properties reflecting the 12-site flip in 2D.

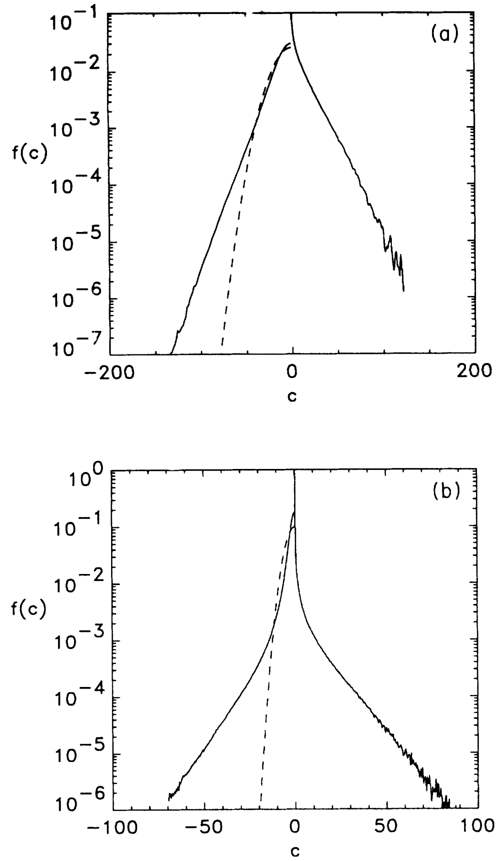

FIG. 2. Semilogarithmic plot of the PDF of c for advection diffusion in 1D based on flips with a half-Gaussian distribution of sizes. Left of center line: simulation results of HP [5] (solid); Gaussian of the same variance (dashed). Right of center line: predictions of Langevin theory (solid). (a) $K = 0.6536$; (b) $K = 13.07$.

For this process, the PDF of the deviations η_j of Eq. (17) is

$$g(\eta) = \tfrac{1}{12}[2\delta(\eta-2) + 3\delta(\eta-1) + 2\delta(\eta) + 3\delta(\eta+1) + 2\delta(\eta+2)], \tag{25}$$

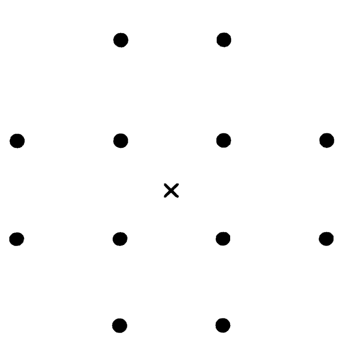

FIG. 3. Array of sites subject to the 12-site flip. The flip consists of 90° clockwise or counterclockwise rotation of the array about its center, denoted $\times$.

reflecting the number of jumps within the 12-site flip corresponding to changes in i of 0, ± 1, and ± 2, respectively. [Note that j in Eq. (17) is an event index, while in Eq. (24) it is a lattice coordinate.] The statistics of the time separations s_j in Eq. (18) are again governed by the exponential distribution with mean T. In terms of the flip frequency per lattice site (ρa^2 in HP notation), the mean time between events affecting a given site is $T = 1/(12\rho a^2)$.

In 2D, $\beta = 4\kappa/L^2$, reflecting the factor of 4 in Eq. (24). Following HP, L and ξ are set equal to $2a$. (As earlier, this is somewhat arbitrary.) Based on these relations, $K = \kappa/(16\rho a^4)$ and $\beta T = 4K/3$.

Langevin theory predictions for this process are compared to simulation results of HP in Fig. 4. As noted by HP, the results indicate that that a change of spatial dimension does not fundamentally impact the statistics of the process. The double shoulder reflects the different types of jumps within the 12-site flip. Langevin theory captures the qualitative features. The quantitative accuracy is also noteworthy.

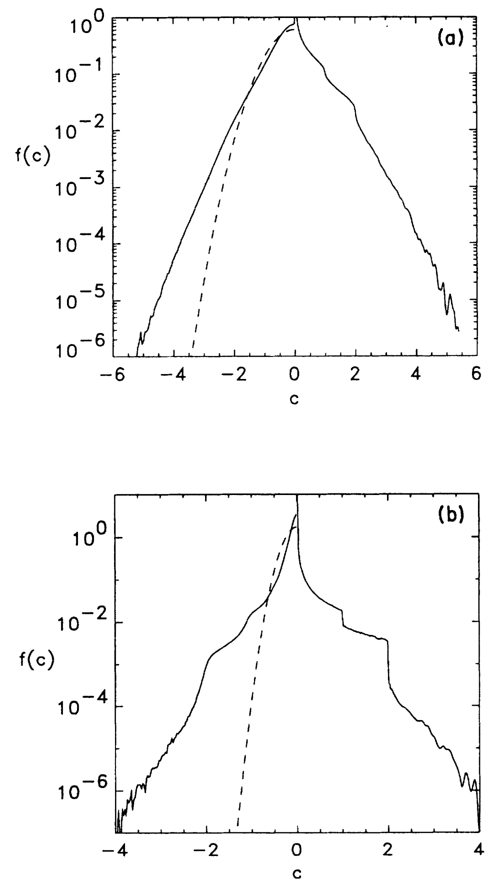

FIG. 4. Semilogarithmic plot of the PDF of c for advection diffusion in 2D based on the 12-site flip of Fig. 3. Left of center line: simulation results of HP [5] (solid); Gaussian of the same variance (dashed). Right of center line: predictions of Langevin theory (solid). (a) $K = 1.66$; (b) $K = 16.6$.

V. DISCUSSION

Previous mean-field analysis of random advection was motivated by experimental and computational results indicating the occurrence of long-tailed scalar PDF's in turbulence. Here, it has been shown that this behavior is not a generic prediction of mean-field theory. Rather, the tail shape predicted by mean-field theory is found to be sensitive to the advection process. In particular, the present analysis indicates that exponential falloff inferred from previous mean-field analysis is a consequence of a modeling assumption that, in effect, selects exponential statistics. This assumption is avoided here by means of an exact treatment of advection. The present approach reveals a wider range of possible behaviors within the framework of mean-field theory than had previously been supposed.

For jump-type advection processes, the present approach predicts Poisson-like PDF tails. PDF tails for jump-type processes may not be of exactly this form [5], but the distinction between this form, exponential tails, or some intermediate behavior is undetectable at the precision of practical computations and experiments. Thus, the agreement that is obtained between mean-field predictions of tail behavior and simulation results for jump-type advection does not guarantee the mechanistic validity of the present mean-field analysis.

The exact treatment of advection in the present approach allows data comparisons to be interpreted in terms of the approximate treatment of the diffusion process. Two alternative approximations are adopted that differ both in physical interpretation and in the mathematical development that ensues. Generalized HP theory is based on the assumed statistical independence of fluid elements separated by a characteristic advection distance. Langevin theory is ostensibly more simplistic in that it neglects entirely the deviations from mean properties in the neighborhood of a given fluid element, and in this sense is more literally a mean-field theory. Nevertheless, Langevin theory is found to be in better agreement with simulation results. It captures features of scalar PDF sensitivity to the relative strength of advection versus diffusion, and to details of the advection process. Significant discrepancies are found, reflecting limitations of the present formulation.

The Langevin theory is amenable to generalizations that may address these limitations. In particular, the decay parameter β representing diffusion effects is held fixed in the present formulation, although theoretical analysis [7] indicates that its fluctuation properties may have significant impact on PDF shape. The present modeling framework does not provide a characterization of the fluctuations of β. The Lagrangian path-integral approach [7], which incorporates a more detailed representation of the interaction of a fluid element with its surroundings, may provide such a characterization. Its application to jump-type advection might clarify the similarities and differences between this type of advection and continuum advection, and might stimulate further development of mean-field theory.

In the latter regard, it is interesting to note parallels between the mean-field theories developed here and two widely used engineering models of turbulent mixing, the coalescence-dispersion (CD) and interaction by exchange-with-the-mean (IEM) models [10]. Both models can be cast in terms of Eq. (4) with the random forcing term omitted. Without the random forcing, scalar fluctuations are not maintained, and the PDF of c evolves to a fully mixed state. Within CD, this evolution is governed by the convolution of the (now time-varying) scalar PDF with itself, as in generalized HP theory. Within IEM, the evolution is governed by exponential decay toward the fully mixed state, as in Langevin theory. Within either model, changes in the local environment are represented by incorporating empirical source and sink terms.

The key distinction between these models and the present mean-field theories is that the source of fluctuations in the mean-field theories is an exact representation of advective effects in the imposed-scalar-gradient configuration. It remains to be determined whether the further development of mean-field theory can contribute to the improvement of engineering models applied to more general configurations.

ACKNOWLEDGMENTS

The authors thank M. Holzer for providing the simulation results used in data comparisons. This research was supported by the Division of Engineering and Geosciences, Office of Basic Energy Sciences, U.S. Department of Energy, and by the National Science Foundation under Grant No. CTS9258445.

[1] A. Pumir, B. I. Shraiman, and E. D. Siggia, Phys. Rev. Lett. **66**, 2984 (1991).
[2] Jayesh and Z. Warhaft, Phys. Rev. Lett. **67**, 3503 (1991); Phys. Fluids A **4**, 2292 (1991).
[3] J. Gollub, J. Clarke, M. Gharib, B. Lane, and O. Mesquita, Phys. Rev. Lett. **67**, 3507 (1991).
[4] A. R. Kerstein, J. Fluid Mech. **231**, 261 (1991).
[5] M. Holzer and A. Pumir, Phys. Rev. E **47**, 202 (1993).
[6] E. S. C. Ching and Y. Tu (unpublished).
[7] B. I. Shraiman and E. D. Siggia, Phys. Rev. E (to be published).
[8] M. Holzer and E. D. Siggia, Phys. Fluids A **5**, 2525 (1993).
[9] H. Takayasu and Y-h. Taguchi, Phys. Rev. Lett. **70**, 782 (1993).
[10] J. Villermaux, in *Encyclopedia of Fluid Mechanics, Vol. 2* (Gulf, Houston, 1986), p. 707.
[11] M. F. Shlesinger, B. J. West, and J. Klafter, Phys. Rev. Lett. **58**, 1100 (1987).
[12] S. S. Wilks, *Mathematical Statistics* (Wiley, New York, 1962).